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A finite difference method for singularly perturbed differential-difference equations arising from a model of neuronal variability

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Abstract

In this paper a finite difference method is presented for singularly perturbed differential-difference equations with small shifts of mixed type (i.e., terms containing both negative shift and positive shift). Similar boundary value problems are associated with expected first exit time problems of the membrane potential in the models for the neuron. To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger to perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior or oscillatory behavior of the solution of the problem.

Keywords: Differential-difference equation; Singular perturbations; Boundary layer; Oscillations; Finite difference method

1. Introduction

Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. An over view of some existence and uniqueness results and applications of singularly perturbed equations may be obtained in [1–4]. Various approaches to the design and analysis of approximate numerical methods for singularly perturbed differential equations can be found in [5–8] and the references cited in them.

In recent years, there has been a growing interest in the numerical study of singularly perturbed

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differential–difference equations because of their applications in many scientific and technical fields [9 and references there in]. A singularly perturbed differential–difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one shift or delay term. The determination of the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites can be modeled as a first-exit time problem. The case of inputs distributed as a Poisson process with exponential decay between the inputs was formulated by Stein [10] and studied by Tuckwell [11,12], and by Wilbur and Rinzel [13]. If, in addition, there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for the expected first-exit time y, given the initial membrane potential $x \in (x_1, x_2)$, can be formulated as a general boundary-value problem for the linear second-order differential-difference equation

$$\frac{\sigma^2}{2} y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I)y(x) = -1,$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. The first-order derivative term $-xy'$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as Poisson processes with mean rates λ_E and λ_I respectively, and produce jumps in the membrane potential of amounts a_E and a_I, respectively, which are small quantities and could depend on voltage. The boundary condition is $y(x) = 0$, $x \notin (x_1, x_2)$. The singular perturbation analysis of boundary-value problem for differential–difference equations with small shifts has been given by Lange and Miura [14,15]. In recent years, there has been a growing interest in the numerical study of such problems owing to its applications in areas such as neurobiology [14], optimal control theory [16,17], in the study of an optically bistable devices [18], in describing the human pupil-light reflex [19], in variety of models for physiological processes or diseases [20–22]. The numerical study of second order singularly perturbed differential–difference equation with small shift or delay has been given in [23–37] and references therein. Amiraliyev and Cimen [38] have given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term.

In this paper a finite difference method is presented for singularly perturbed differential–difference equations with small shifts of mixed type (i.e., terms containing both negative shift and positive shift). To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger than the perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior or oscillatory behavior of the solution of the problem.

2. Finite difference method

To describe the method, we consider a linear singularly perturbed differential–difference equation of mixed type i.e., equation containing both the negative and positive shift terms

$$\varepsilon^2 y''(x) + \alpha(x)y(x - \delta) + \omega(x)y(x) + \beta(x)y(x + \eta) = f(x) \tag{1}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the interval and boundary conditions

$$\begin{aligned} y(x) &= \phi(x), \quad -\delta \le x \le 0 \\ y(x) &= \psi(x), \quad 1 \le x \le 1 + \eta \end{aligned} \tag{2}$$

where $\alpha(x)$, $\omega(x)$, $\beta(x)$, $f(x)$, $\phi(x)$ and $\psi(x)$ are known analytic functions and, further, that each function is simple enough so that analytic differentiation is feasible, δ and η are the small shifting parameters. For the shifts δ, η equal to zero and if $\alpha(x) + \omega(x) + \beta(x) < 0$ on the interval [0,1], then the solution exhibits boundary layers at both the ends of the interval [0,1]. The novelty of this class of equations is that the negative and positive shifts δ, η respectively need not be same. In this paper we study the effects of the nonzero shifts on the boundary layer solution.

To develop a finite difference method, we construct a special type of mesh so that the terms containing the shift parameters lie on the nodal points after discretization. We divide the interval [0,1] into N equal parts by choosing the mesh parameter h such that $h = \delta/k = \eta/l$, where k and l are positive integers chosen such that $1 \le k$, $1 \le N$.

Using central difference formulae, the finite difference representation of Eq. (1) may be written at a typical mesh point x_i, $i=0, 1, \ldots, N$, as

$$\frac{\varepsilon^2}{h^2}\left\{\delta^2 - \frac{1}{12}\delta^4\right\} y_i + \alpha_i y_{i-k} + \omega_i y_i + \beta_i y_{i+\ell} = f_i + B_i y_i \tag{3}$$

where

$$B_i = -\frac{\varepsilon^2}{h^2}\left\{\frac{1}{90}\delta^6 + \ldots\right\} \tag{4}$$

and $\alpha(x_i)=\alpha_i$; $\omega(x_i)=\omega_i$; $\beta(x_i)=\beta_i$; $f(x_i)=f_i$; $y(x_i)=y_i$ and $\boldsymbol{\delta}$ is the standard central difference operator.

The boundary conditions become

$$\begin{aligned} y_i &= \phi_i, \quad \text{for} \quad -k \le i \le 0 \\ y_i &= \psi_i; \quad \text{for} \quad N \le i \le N+l \end{aligned} \tag{5}$$

where

$$\phi_i = \phi(x_i), \qquad \psi_i = \psi(x_i).$$

Differentiating (1) twice, then using central difference formulae, gives a $O(h^6)$ approximation for $\boldsymbol{\delta}^4 y_i$ as follows [c.f. Boqucz and Walker [39]]:

$$\begin{aligned} \frac{\varepsilon^2}{h^2}\boldsymbol{\delta}^4 y_i = {} & h^2 f_i'' - \alpha_i \boldsymbol{\delta}^2 y_{i-k} - \omega_i \boldsymbol{\delta}^2 y_i - \beta_i \boldsymbol{\delta}^2 y_{i+\ell} \\ & - 2\alpha_i' h \boldsymbol{\mu\delta} y_{i-k} - 2\omega_i' h \boldsymbol{\mu\delta} y_i - 2\beta_i' h \boldsymbol{\mu\delta} y_{i+\ell} \\ & - \alpha_i'' h^2 y_{i-k} - \omega_i'' h^2 y_i - \beta_i'' h^2 y_{i+\ell} + C_i y_i \end{aligned} \tag{6}$$

where$C_i = \frac{\varepsilon^2}{6h^4}\left\{\boldsymbol{\delta}^6 + \ldots\right\}$ and $\boldsymbol{\mu}$ is the averaging operator.

Substituting (6) in (3) and simplifying, we get the finite difference scheme as

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^* y_{i-k-1} + F_i^* y_{i-k} \\ + G_i^* y_{i-k+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^{**} y_{i+\ell+1} = R_i + \tilde{E}_i y_i \end{aligned} \tag{7}$$

where

$$E_i = \frac{\varepsilon^2}{h^2} + \frac{\omega_i}{12} - \frac{\omega_i' h}{12},$$

$$F_i = -\frac{2\varepsilon^2}{h^2} - \frac{\omega_i}{6} + \frac{\omega_i''}{12}h^2 + \omega_i,$$

$$G_i = \frac{\varepsilon^2}{h^2} + \frac{\omega_i}{12} + \frac{\omega_i'}{12}h,$$

$$E_i^* = \frac{\alpha_i}{12} - \frac{\alpha_i'}{12}h,$$

$$F_i^* = -\frac{\alpha_i}{6} + \frac{\alpha_i''}{12}h^2 + \alpha_i,$$

$$G_i^* = \frac{\alpha_i}{12} + \frac{\alpha_i'}{12}h,$$

$$E_i^{**} = \frac{\beta_i}{12} - \frac{\beta_i'}{12}h,$$

$$F_i^{**} = -\frac{\beta_i}{6} + \frac{\beta_i''}{12}h^2 + \beta_i,$$

$$G_i^{**} = \frac{\beta_i}{12} + \frac{\beta_i'}{12}h, \; R_i = f_i + \frac{h^2}{12}f_i'',$$

$$\tilde{E}_i = \frac{\varepsilon^2}{6h^2}\left(-\frac{1}{15} + \frac{1}{12h^2}\right)\delta^6 + \ldots$$

By using the boundary conditions (5), the above difference scheme (7) can be rewritten as

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^{**} y_{i+\ell+1} = R_i - E_i^* \phi_{i-k-1} - F_i^* \phi_{i-k} \\ - G_i^* \phi_{i-k+1} \qquad \text{for} \quad 1 \le i \le k-1 \end{aligned}$$

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} = R_i - E_i^* \phi_{i-k-1} \\ - F_i^* \phi_{i-k} \qquad \text{for} \quad i = k \end{aligned}$$

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} = R_i \\ - E_i^* \phi_{i-k-1} \qquad i = k+1 \end{aligned}$$

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} \\ + E_i^* y_{i-k-1} = R_i \qquad \text{for} \;\; k+2 \le i \le N \\ - \ell - 2 \end{aligned}$$

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} \\ + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i \\ - G_i^{**} \psi_{i+\ell+1} \qquad \text{for} \quad i = N - \ell - 1 \end{aligned}$$

$$\begin{aligned} E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + G_i^* y_{i-k+1} \\ + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} \\ - F_i^{**} \psi_{i+\ell} \qquad \text{for} \quad i = N - \ell \end{aligned}$$

Table 1
Maximum absolute errors for example 1 when $\delta = 0.03$.

$\varepsilon\downarrow$	$N\rightarrow$	100	200	300	400	500
2^{-1}		6.4002e−006	1.6002e−006	7.1122e−007	4.0007e−007	2.5604e−007
2^{-2}		4.8102e−005	1.2030e−005	5.3470e−006	3.0077e−006	1.9250e−006
2^{-3}		3.3978e−004	8.5055e−005	3.7811e−005	2.1271e−005	1.3614e−005
2^{-4}		2.2148e−003	5.5625e−004	2.4743e−004	1.3922e−004	8.9114e−005
2^{-5}		1.2858e−002	3.2670e−003	1.4564e−003	8.2011e−004	5.2513e−004
2^{-6}		6.5377e−002	1.7359e−002	7.8046e−003	4.4079e−003	2.8264e−003

Table 2
Maximum absolute errors for example 2 when $\delta = 0.03$.

$\varepsilon\downarrow$	$N\rightarrow$	100	200	300	400	500
2^{-1}		6.3794e−006	1.5950e−006	7.0889e−007	3.9876e−007	2.5520e−007
2^{-2}		4.8620e−005	1.2159e−005	5.4043e−006	3.0400e−006	1.9456e−006
2^{-3}		3.4642e−004	8.6711e−005	3.8547e−005	2.1684e−005	1.3878e−005
2^{-4}		2.2640e−003	5.6851e−004	2.5288e−004	1.4228e−004	9.1074e−005
2^{-5}		1.3156e−002	3.3419e−003	1.4897e−003	8.3886e−004	5.3713e−004
2^{-6}		6.7055e−002	1.7797e−002	8.0010e−003	4.5187e−003	2.8974e−003

$$\begin{aligned} E_i y_{i-1} &+ F_i y_i + G_i y_{i+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} \\ &+ E_i^* y_{i-k-1} = R_i - G_i^{**}\psi_{i+\ell+1} - F_i^{**}\psi_{i+\ell} \\ &- E_i^{**}\psi_{i+\ell-1} \quad for \quad N-\ell+1 \le i \le N-1 \end{aligned}$$

The above system of equations along with the boundary conditions $y_0 = \phi_0$ and $y_N = \psi_N$ is solved for y_i, $i = 0, 1, 2, \ldots, N$ by Gauss elimination method with partial pivoting. In fact, any numerical method or analytical method can be used to solve the above system of equations for y_i.

3. Numerical results

To demonstrate the applicability of the method we consider boundary value problems of singularly perturbed linear differential difference equations exhibiting boundary layers at both sides of the underlying interval [0,1]. These examples were discussed in Refs. [14,15,25]. Since the exact solutions of the problems for different values of δ and η are not known, the maximum absolute errors for the examples are calculated using the double mesh principle

$$e_N = \max_{0\le i\le N} |y_i^N - y_{2i}^{2N}| [\text{c.f. } 39]$$

we discuss the numerical examples in three cases.

Case I: when $\beta(x) = 0$

In this case, problems (1) and (2) reduces to

$$\varepsilon^2 y''(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) = f(x) \tag{8}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the interval and boundary conditions

$$y(x) = \phi(x), \qquad -\delta \le x \le 0$$
$$y(1) = \psi$$

where ψ is constant.

For this case, the scheme given in (7) can be modified accordingly by putting $\beta_i = 0$. The system of equations along with the boundary conditions $y_0 = \phi_0$ and $y_N = \psi$ is solved for y_i, $i = 0, 1, 2, \ldots, N$ by Gauss elimination method with partial pivoting.

Example 1. [15, p. 282]: $\varepsilon^2 y''(x) - 2e^{-x}y(x-\delta) - y(x) = 1$, subject to the interval and boundary conditions $y(x) = 1\,;\ -\delta \le x \le 0$, $y(1) = 0$.

Example 2. [14, p. 263]: $\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) = 1$ subject to the interval and boundary conditions $y(x) = 1\,;\ -\delta \le x \le 0$, $y(1) = 0$.

The maximum absolute errors for $\delta = 0.03$ for different values of ε are presented in Tables 1 and 2 respectively for examples 1 and 2. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. Figs. 1 and 5 show the effect of shift parameter δ on the boundary layer solution for examples 1 and 2 respectively, when shift parameter δ is smaller than perturbation parameter ε. Figs. 2–4 and 6–8 show the effect of shift parameter δ on the boundary layer solution for examples 1

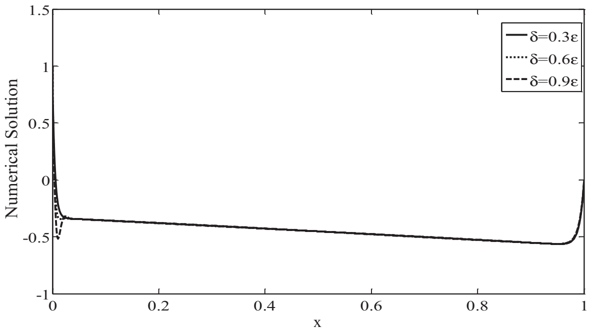

Fig. 1. Numerical solution of example 1 with $\varepsilon = 0.01$ for different values of δ.

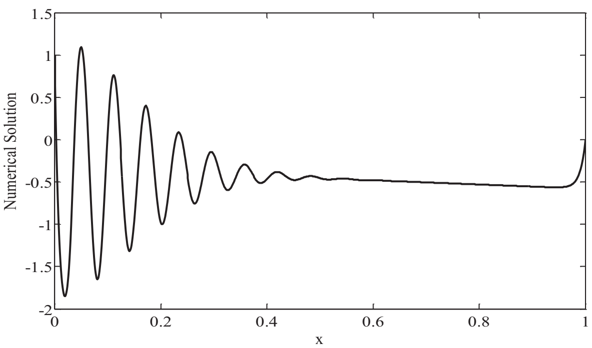

Fig. 2. Numerical solution of example 1 with $\varepsilon = 0.01$ and $\delta = 0.03$.

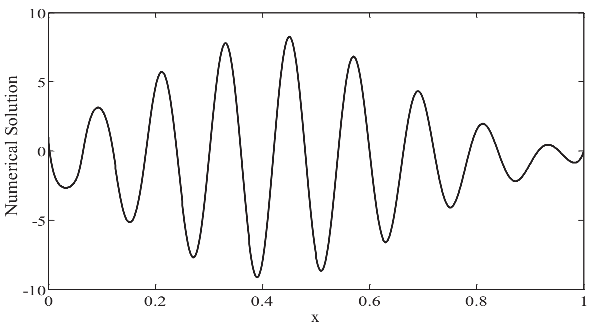

Fig. 3. Numerical solution of example 1 with $\varepsilon = 0.01$ and $\delta = 0.06$.

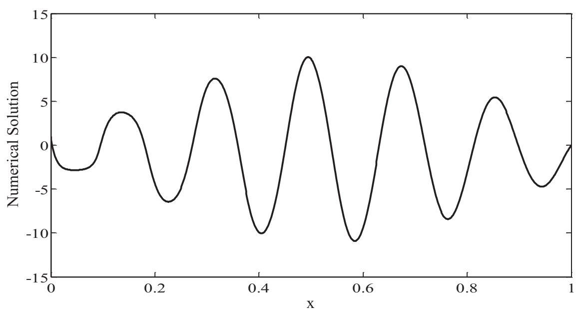

Fig. 4. Numerical solution of example 1 with $\varepsilon = 0.01$ and $\delta = 0.09$.

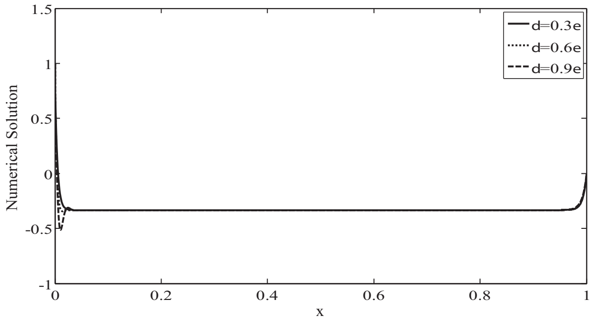

Fig. 5. Numerical solution of example 2 with $\varepsilon = 0.01$ for different values of δ.

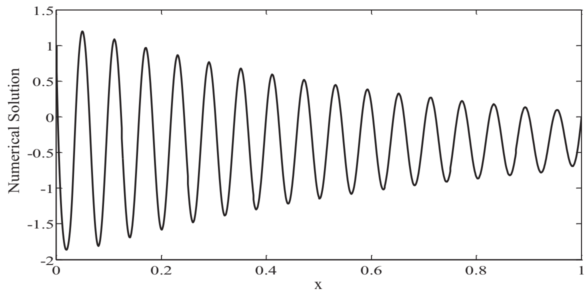

Fig. 6. Numerical solution of example 2 with $\varepsilon = 0.01$ and $\delta = 0.03$.

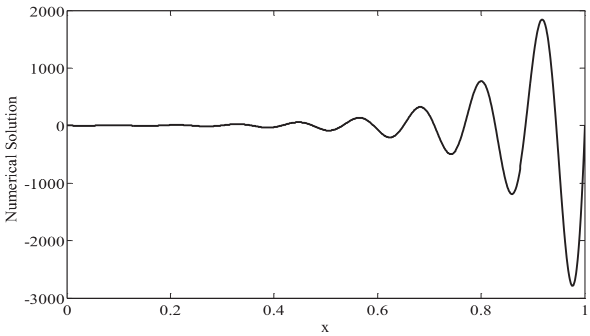

Fig. 7. Numerical solution of example 2 with $\varepsilon = 0.01$ and $\delta = 0.06$.

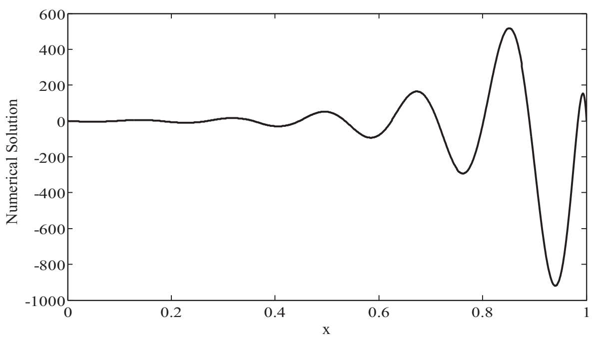

Fig. 8. Numerical solution of example 2 with $\varepsilon = 0.01$ and $\delta = 0.09$.

Table 3
Maximum absolute errors for example 3 when $\eta = 0.03$.

$\varepsilon\downarrow$	$N\rightarrow$	100	200	300	400	500
2^{-1}		1.1834e−006	2.9587e−007	1.3150e−007	7.3969e−008	4.7340e−008
2^{-2}		1.2028e−005	3.0078e−006	1.3369e−006	7.5202e−007	4.8130e−007
2^{-3}		8.6603e−005	2.1677e−005	9.6364e−006	5.4209e−006	3.4695e−006
2^{-4}		5.6601e−004	1.4213e−004	6.3220e−005	3.5571e−005	2.2769e−005
2^{-5}		3.2890e−003	8.3549e−004	3.7244e−004	2.0972e−004	1.3428e−004
2^{-6}		1.6763e−002	4.4494e−003	2.0002e−003	1.1296e−003	7.2435e−004

and 2 respectively, when shift parameter δ is larger than perturbation parameter ε.

Case II: when $\alpha(x) = 0$

In this case, problems (1) and (2) reduces to

$$\varepsilon y''(x) + \omega(x)y(x) + \beta(x)y(x+\eta) = f(x) \tag{9}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the interval and boundary conditions

$$y(0) = \phi$$
$$y(x) = \psi(x), \qquad 1 \le x \le 1+\eta$$

where ϕ is constant.

For this case, the scheme given in (7) can be modified accordingly by putting $\alpha_i = 0$. The system of equations along with the boundary conditions $y_0 = \phi$ and $y_N = \psi_N$ is solved for y_i, $i = 0, 1, 2, \ldots, N$ by Gauss elimination method with partial pivoting.

Example 3. [25, p. 158]: $\varepsilon^2 y''(x) - y(x) - 2y(x+\eta) = 1$, subject to the interval and boundary conditions $y(0) = 1$, $y(x) = 0$, $1 \le x \le 1+\eta$.

The maximum absolute errors for $\eta = 0.03$ for different values of ε are presented in Table 3 for example 3. From table, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. Fig. 9

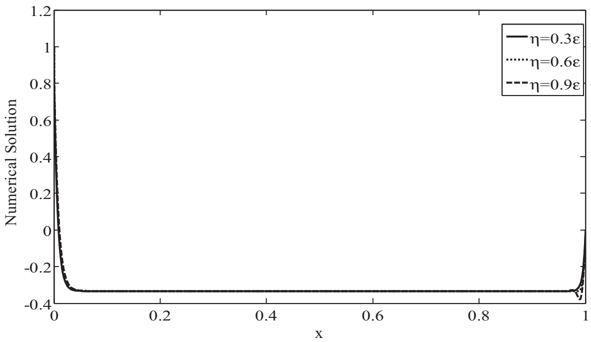

Fig. 9. Numerical solution of example 3 with $\varepsilon = 0.01$ for different values of η.

shows the effect of shift parameter η on the boundary layer solution for example 3, when shift parameter η is smaller than perturbation parameter ε. Figs. 10–12 show the effect of shift parameter η on the boundary layer solution for example 3, when shift parameter η is larger than perturbation parameter ε.

Case III: when $\alpha(x) \neq 0$, $\beta(x) \neq 0$

In this case we consider numerical examples in which both the negative and positive shifts occur. The system of equations (7) along with the boundary conditions $y_0 = \phi_0$ and $y_N = \psi_N$ is solved for y_i, $i = 0, 1, 2, \ldots, N$ by Gauss elimination method with partial pivoting.

Example 4. [14,p. 265]: $\varepsilon^2 y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1$, subject to the interval conditions $y(x) = 1\,;\ -\delta \le x \le 0$, $y(x) = 0\,;\ 1 \le x \le 1+\eta$.

Example 5. [14,p. 265]: $\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) - 2y(x+\eta) = 1$ subject to the interval conditions $y(x) = 1\,;\ -\delta \le x \le 0$, $y(x) = 0\,;\ 1 \le x \le 1+\eta$.

The maximum absolute errors for $\delta = 0.03$, $\eta = 0.07$ for different values of ε are presented in Table 4 for example 4. The maximum absolute errors for $\delta = 0.07$, $\eta = 0.03$ for different values of ε are presented in Table 5 for example 5. From the tables, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. Figs. 13 and 14 show the effect of shift parameters δ and

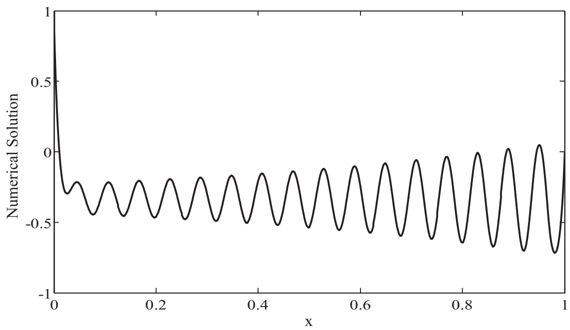

Fig. 10. Numerical solution of example 3 with $\varepsilon = 0.01$ and $\eta = 0.03$.

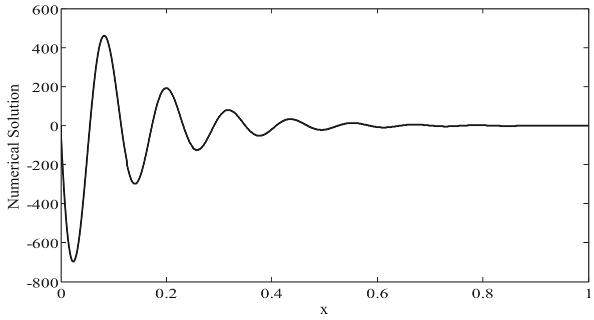

Fig. 11. Numerical solution of example 3 with $\varepsilon = 0.01$ and $\eta = 0.06$.

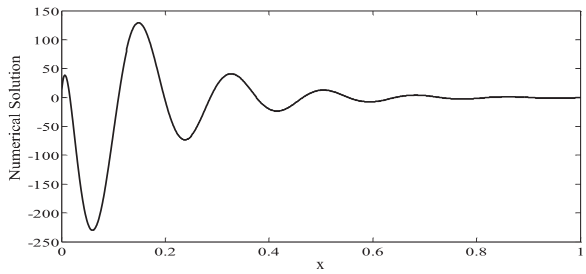

Fig. 12. Numerical solution of example 3 with $\varepsilon = 0.01$ and $\eta = 0.09$.

η on the boundary layer solution for example 4, when shift parameters δ and η are smaller than perturbation parameter ε. Fig. 15 shows the effect of shift parameters on the boundary layer solution for example 5, when δ is larger than ε and η is smaller than ε. Fig. 16 shows the effect of shift parameters on the boundary layer solution for example 5, when η is larger than ε and δ is smaller than ε.

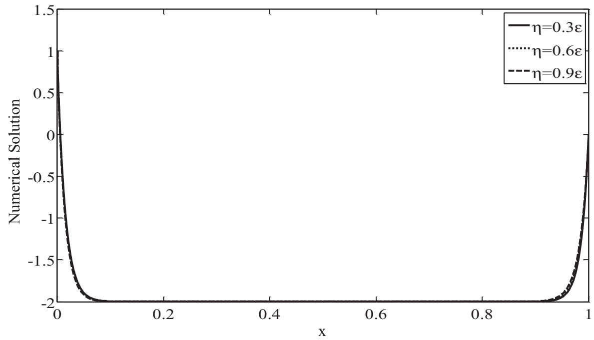

Fig. 13. Numerical solution of example 4 with $\varepsilon = 0.01$ and $\delta = 0.005$ for different values of η.

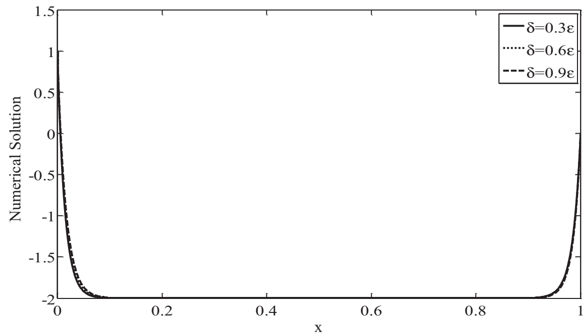

Fig. 14. Numerical solution of example 4 with $\varepsilon = 0.01$and $\eta = 0.005$ for different values of δ.

Table 4
Maximum absolute errors for example 4 when $\delta = 0.03$, $\eta = 0.07$.

$\varepsilon\downarrow$	$N\rightarrow$	100	200	300	400	500
2^{-1}		6.0760e−007	1.5154e−007	6.7352e−008	3.7886e−008	2.4246e−008
2^{-2}		6.7402e−006	1.6856e−006	7.4919e−007	4.2143e−007	2.6972e−007
2^{-3}		5.0777e−005	1.2709e−005	5.6498e−006	3.1783e−006	2.0342e−006
2^{-4}		2.9686e−004	7.4635e−005	3.3206e−005	1.8685e−005	1.1961e−005
2^{-5}		1.6272e−003	4.1624e−004	1.8578e−004	1.0465e−004	6.7025e−005
2^{-6}		6.6542e−003	1.8089e−003	8.1653e−004	4.6180e−004	2.9630e−004
2^{-7}		2.0534e−002	6.6398e−003	3.1032e−003	1.7770e−003	1.1467e−003

Table 5
Maximum absolute errors for example 5 when $\delta = 0.07$, $\eta = 0.03$.

$\varepsilon\downarrow$	$N\rightarrow$	100	200	300	400	500
2^{-1}		1.4505e−005	3.6266e−006	1.6119e−006	9.0669e−007	5.8028e−007
2^{-2}		9.3649e−005	2.3423e−005	1.0411e−005	5.8563e−006	3.7481e−006
2^{-3}		5.2694e−004	1.3195e−004	5.8664e−005	3.3002e−005	2.1122e−005
2^{-4}		2.5668e−003	6.4571e−004	2.8731e−004	1.6168e−004	1.0349e−004
2^{-5}		9.9696e−003	2.5626e−003	1.1448e−003	6.4512e−004	4.1322e−004
2^{-6}		3.1132e−002	8.7328e−003	3.9629e−003	2.2454e−003	1.4418e−003
2^{-7}		8.7650e−002	2.8308e−002	1.3612e−002	7.8739e−003	5.1051e−003

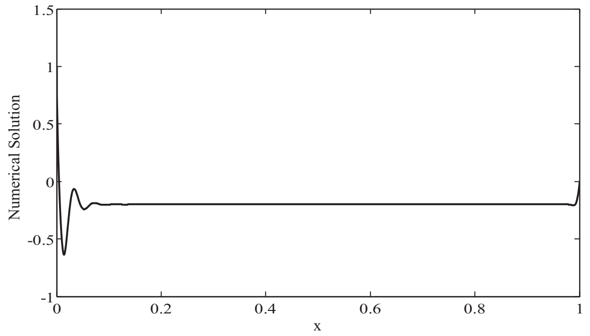

Fig. 15. Numerical solution of example 5 with $\varepsilon = 0.01$ and $\delta = 0.015$, $\eta = 0.007$.

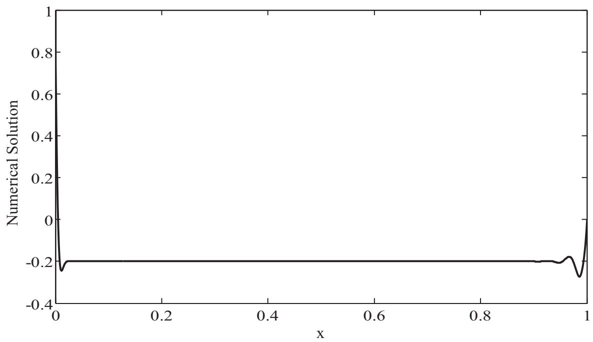

Fig. 16. Numerical solution of example 5 with $\varepsilon = 0.01$ and $\delta = 0.007$, $\eta = 0.015$.

4. Conclusions

Boundary value problems for linear second order singularly perturbed differential–difference equations with small shifts of mixed type (i.e., terms containing both negative shift and positive shift) are considered. To obtain an approximate solution for such type of boundary value problems a finite difference method is presented. To handle the shift parameters, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization and a fourth order finite difference method is applied. The proposed finite difference method works nicely when the delay parameter is smaller or bigger to perturbation parameter. The proposed finite difference method is conceptually simple, easy to use, and readily adapted for computer implementation with modest amount of problem preparation. The maximum absolute error is tabulated in the form of Tables 1–5 for the considered examples in support of the predicted theory. The graphs of the solution of the considered examples for different values of shift parameter are plotted in Figs. 1–16 to examine the effect of shift on the boundary layer and oscillatory behavior of the solution.

It is observed that when the delay parameter is smaller than the perturbation parameter and in the absence of positive shift term, as δ increases, the thickness of the left boundary layer decreases and that of the right boundary layer increases. If there is only a positive shift, then as η increases, the thickness of the left boundary layer increases and that of the right boundary layer decreases. If there are both the positive and negative shifts present, then by varying either δ or η, the thickness of the left boundary layer decreases and that of the right boundary layer increases and for shift parameters greater than the perturbation parameter, it is observed that the layer behavior of the solution is no longer maintained and the solution exhibits oscillatory behavior. Also when the shifts further increase the oscillations previously confined to the layer region are extended throughout the interval [0,1]. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. On the basis of the extensive numerical work, it is concluded that the present method offers significant advantage for the linear singularly perturbed differential difference equations of mixed type.

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